Hydrogen in Electrodynamics. I. Preliminary Theories

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Z. Naturforsch. **43a**, 1039–1043 (1988); received October 5, 1988

After a discussion of the one-component Schrödinger (1926) and the four-component Dirac (1928) representation of hydrogen it is shown that the six-component electrodynamic picture turns out to be considerably simpler and clearer. The computational effort is reduced to a fraction.

We start from the fact that source-free electrodynamics and relativistic wave mechanics are formally identical theories [1]. Wave mechanics yields the, so far, most useful hydrogen model. Because of the above identity electrodynamics must be able to do so likewise. Even more: It can be expected that the electrodynamic model turns out to be considerably more informative, since a full six-component clearness takes the place of a four-component probability version.

Following the historical development we recall in this paper first the mechanical model, then the one-component Schrödinger model, further the four-component Dirac model and finally have a preliminary look at the six-component electrodynamic model.

1. The Mechanical Hydrogen Model

A straight line, which is held fixed in a point and rotates circularly, generates a circular cone. This represents the simplest surface which can be generated by the three most elementary geometric forms – point, straight line and circle. If we investigate the cone by intersecting it with a plane in any manner then we obtain the conic sections, the orbits of the Kepler systems. By partitioning the energy, then immediately their unambiqvous connection with the hydrogen spectrum becomes apparent. Thereby it does not matter how large the two masses of the system, absolutely as well as relatively, have been chosen, as long as we are free to choose the dimensions of the orbits and specify the potential of the inverse distance. In other words: the space-time structure of the hydrogen atom has a direct and unambiguous connection with the elementary geometric forms – point, straight line and circle.

After Rutherford had sensed the Kepler-properties of hydrogen in 1911 and had supported his ideas by the well-known scattering experiments, Bohr, in 1913, found the unambiguous analogy between Kepler motion and hydrogen spectrum by quantizing the energy according to Planck. In 1915 Sommerfeld brought the mechanical model to its final form by restricting the velocity according to Einstein and thus obtained the well-known relation for the frequencies

$$\nu^{\mathrm{hyd}} = \frac{m_0 c^2}{h\sqrt{1+\dfrac{\alpha^3}{(n_r+\sqrt{k^2-\alpha^2})^2}}}, \tag{1}$$

which, in view of its efficiency, one would characterize as final. Heisenberg incorporated the Kepler system into his matrix mechanics, which, however, turned out to be rather cumbersome in its application to, e.g., the velocity-restricted model. The uncertainty relations, from the point of view of electrodynamics. seem not to refer to particles but to the barycenters of wave fields.

2. The One-Component Hydrogen Model

In his search for a differential description of de Broglie's waves [2] Schrödinger [3] in 1926 inserted Hamilton's analogy [4]

"A body in a potential behaves like light in a medium, whereby the following connection exists between potential and index of refraction

$$N = \frac{c\sqrt{2m(U-\Phi)}}{U}\ " \tag{2}$$

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into the classical wave equation for light

$$\left(\Delta - \frac{N^2}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi = 0, \tag{3}$$

and thus obtained wave mechanics

$$\left[\Delta - 2m\left(\frac{\Phi}{\hbar^2} - \frac{i}{\hbar}\frac{\partial}{\partial t}\right)\right]\Psi = 0. \tag{4}$$

Wave mechanics, therefore, basically is a theory of light-fields in the index of refraction (2). Dropping the assumption of an infinitely heavy central mass, which is usually made in the wave-mechanical hydrogen model, and considering more generally a two-body system, then Hamilton's analogy, extrapolated to two masses, would read:

> Two interacting masses behave like two interacting light-fields, whereby the connection (2) exists between potential and index of refraction.

Let us question for the moment the finality of the Copenhagen interpretation and the supporting claim of Rutherford's scattering experiments and envisage the hydrogen atom as a system of two light-fields.

We imagine the two interacting, respectively mutually refracting, light-fields, in view of the exponential form of the hydrogen Schrödinger function

$$\Psi^{\text{hyd}} = C\,R\,P\,e^{im\varphi}\,e^{i\omega t} = \Psi^{\text{Re}} + i\,\Psi^{\text{Im}}, \tag{5}$$

as two wave fields of the same amplitude but different phase, whereby one of the light-fields shows itself as real and the other one as imaginary. Here it should be especially noticed that both branches of the solution, the real as well as the imaginary, should represent a light field. Thus the complex exponential function will be used here in a way which differs from that in many other parts of electrodynamics. There one uses it, as is well known, mainly because of its computational ease. The process to be described by it will, however, as a rule be read off its real part, its imaginary part being left out of consideration.

We assign to the equal light-fields Ψ^{Re} and Ψ^{Im} equal masses m^{Re} and m^{Im} in the Schrödinger equation, which are connected with the reduced mass m in the well-known way

$$m = \frac{m^{\text{Re}}\,m^{\text{Im}}}{m^{\text{Re}} + m^{\text{Im}}}, \tag{6}$$

from which, because of $m^{\text{Re}} = m^{\text{Im}}$, it follows that

$$m^{\text{Re}} = m^{\text{Im}} = 2m. \tag{7}$$

In order to point out illustrative and comprehensible features of the light-field system we investigate the motion of the centers of energy of the two light-fields. The center of energy of Ψ^{Re} and Ψ^{Im}, under the assumption that both functions are normalized, is given by

$$\bar{\boldsymbol{r}}^{\text{Re}} = \int_\infty (\Psi^{\text{Re}})^2\,\boldsymbol{r}\,\mathrm{d}v, \quad (\boldsymbol{r}\text{: radius vector}) \tag{8}$$

and

$$\bar{\boldsymbol{r}}^{\text{Im}} = \int_\infty (\Psi^{\text{Im}})^2\,\boldsymbol{r}\,\mathrm{d}v. \tag{9}$$

The center of the total energy is obtained as one half of the vector sum of (8) and (9)

$$\bar{\boldsymbol{r}} = \tfrac{1}{2}\int_\infty [(\Psi^{\text{Re}}) + (\Psi^{\text{Im}})^2]\,\boldsymbol{r}\,\mathrm{d}v = \tfrac{1}{2}\int_\infty \Psi\,\overset{*}{\Psi}\,\boldsymbol{r}\,\mathrm{d}v \tag{10}$$

Differentiating twice with respect to time yields

$$2\ddot{\bar{\boldsymbol{r}}} = \frac{\partial^2}{\partial t^2}\int_\infty \Psi\,\overset{*}{\Psi}\,\boldsymbol{r}\,\mathrm{d}v. \tag{11}$$

Ehrenfest's theorem [5] establishes a connection between (11) and the average potential gradient

$$\overline{\text{grad}\,\Phi} = \int_\infty \Psi\,\overset{*}{\Psi}\,\text{grad}\,\Phi\,\mathrm{d}v \tag{12}$$

It states:

$$\frac{\partial^2}{\partial t^2}\int_\infty \Psi\,\overset{*}{\Psi}\,\boldsymbol{r}\,\mathrm{d}v = -\frac{1}{\mathrm{m}}\,\overline{\text{grad}\,\Phi}. \tag{13}$$

From (11) and (13) the equation

$$2m\ddot{\bar{\boldsymbol{r}}} = -\,\overline{\text{grad}\,\Phi} \tag{14}$$

or

$$2m\tfrac{1}{2}(\ddot{\bar{\boldsymbol{r}}}^{\text{Re}} + \ddot{\bar{\boldsymbol{r}}}^{\text{Im}}) = -\,\overline{\text{grad}\,\Phi} \tag{15}$$

follows, and with (7) we get

$$(m^{\text{Re}}\,\ddot{\bar{\boldsymbol{r}}}^{\text{Re}} + \overline{\text{grad}\,\Phi}) + (m^{\text{Im}}\,\ddot{\bar{\boldsymbol{r}}}^{\text{Im}} + \overline{\text{grad}\,\Phi}) = 0. \tag{16}$$

Because of the assumed equality of the masses we finally have

$$m^{\text{Re}}\frac{\partial^2}{\partial t^2}\bar{\boldsymbol{r}}^{\text{Re}} = -\,\overline{\text{grad}\,\Phi} \quad \text{and}$$
$$m^{\text{Im}}\frac{\partial^2}{\partial t^2}\bar{\boldsymbol{r}}^{\text{Im}} = -\,\overline{\text{grad}\,\Phi}. \tag{17}$$

The sketch (5)–(17) can give us an idea about how one could imagine the hydrogen atom to consist of two light-fields.

With (17) this conception furnishes the following model perspectives, whose main features remain of relevance also in the four- and six-component models:

The centers of gravity of two mutually refracting lightwave fields move according to Newton's mechanics.

Since (4) yields the hydrogen spectrum, if and only if the potential has the form

$$\Phi = \text{const}/r, \tag{18}$$

then from (17) it follows furthermore:

The centers of gravity of two interacting light-fields attract each other like two gravitating masses.

Or:

The centers of gravity of two interacting light-fields constitute a Kepler system.

The frequencies of the one-component model, as is well known, depend on the electron mass m_0 (or the reduced mass m, respectively) and the charge e_0 only through the Rydberg number

$$R = \frac{2\pi^2 m_0 e_0^4}{c h^3}. \tag{19}$$

Therefore one may modify m_0 and e_0 within the Rydberg number in such a way that the Rydberg number itself, and thus the frequency, remains unaltered. This, however, means that the one-component model is not tied uniquely to the parameter pair (m_0, e_0), or the Bohr model, respectively. Rather infinitely many pairs of values (m, e) yield the hydrogen frequencies.

Until a later determination – for instance through initial conditions – we therefore could leave the numerical values of m and e undecided, but for the sake of simplicity we shall keep m_0 and e_0 of the Bohr theory further on.

From (17) we furthermore deduce that e has to represent a gravitational parameter in the light-field model, which characterizes the mutual attraction of the centers of gravity of the two light-fields Ψ^{Re} and Ψ^{Im}. The related circumstances in the four-and six-component models will turn out to be in complete analogy.

3. The Four-Component Hydrogen Model

The four-component treatment of the Kepler problem with Dirac necessitates extensive computations where one may easily lose control. According to current assessment the Dirac-hydrogen-model represents the computationally most extensive solution of wave mechanics. Nevertheless, the theory in its present form is not able to express the feature of spin one-half in its terms of the solutions.

Connections between Dirac's theory and Maxwell's equations have been recognized shortly after Dirac's papers. The various attempts at bridging the two theories scatter over half a century [6]–[30]. For most of the time the conviction prevailed that proven connections, as, e.g.,

$$\Psi_1 = i E_3, \quad \Psi_2 = i(E_1 + i E_2), \quad \Psi_3 = H_3,$$
$$\Psi_4 = H_1 + i H_2, \tag{20}$$

are valid only for mass- and force-free systems ($m = 0$, $\Phi = 0$) or for force-free systems ($m \neq 0$, $\Phi = 0$). Textbooks of the fifties and sixties therefore tend to point out that one should not overestimate the remarkable connection (20). In a well-known standard book [29], for instance, we read:

"One should avoid overestimating this result (see (20)). For the relation between matter waves and electromagnetic waves is not as close – or maybe stated better, not as completely transparent – as has been hoped for at the beginning of wave mechanics. If one, e.g., makes the substitution (20) in Dirac's equations for the case of acting forces, and even with $m = 0$, one gets absolutely useless additional terms to Maxwell's equations. The derivation of Maxwell's equations from Dirac's in the field-free case therefore may well be a mathematical coincidence".

Today we know that this is not the case: We rather have seen [30] that source-free electrodynamics and Dirac's equations are isomorphic and therefore Dirac's theory represents a true subtheory of Maxwell's. From this it also follows that the connection (20) is valid not only for mass- and force-free systems ($m = 0$, $\Phi = 0$) but just as well for the general system ($m_0 \neq 0$, $\Phi \neq 0$).

As far as the terms of the field components of hydrogen are concerned, Dirac's theory yields distinctly more than Schrödinger's. The Dirac model already shows different exponentials $e^{im\varphi}$ and $e^{i(m+1)\varphi}$ in the spinor components, but it does not combine them yet. The combination of the two exponentials, however, is the prerequisite for half-integer spin to appear numerically in the field components. In [1], (17), one sees that this combination is reserved to electrodynamics.

4. The Six-Component Hydrogen Model

In order to appreciate properly the drastic clarifying simplification which is brought about by the six-component treatment of the Kepler problem, one should have duplicated first the four-component one in its entirety. – The six-component analysis recommends itself in the first instance because of the recognition [1] that wave mechanics differs from source-free electrodynamics just by a constant matrix factor:

Multiplication of source-free electrodynamics by the Pauli vector yields wave mechanics.

Or in other words: The system of differential equations of source free electrodynamics and relativistic wave mechanics are isomorphic [1]:

$$\underset{\text{(electrodynamics)}}{\left\{\left[\gamma\cdot\nabla + i\frac{\omega}{c}\begin{pmatrix}\varepsilon\mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mu\mathbf{1}\end{pmatrix}\right]\Psi = 0\right\}} \equiv \underset{\text{(wave mechanics)}}{\left\{\left[\gamma\cdot\nabla + i\frac{\omega}{c}\begin{pmatrix}\left(1-\frac{\Phi - m_0c^2}{\hbar\omega}\right)\mathbf{1} & \mathbf{0}\\ \mathbf{0} & \left(1-\frac{\Phi + m_0c^2}{\hbar\omega}\right)\mathbf{1}\end{pmatrix}\right]\Psi = 0\right\}}. \tag{21}$$

Since relativistic wave mechanics contains the hydrogen frequencies, then, because of (21), also source-free electrodynamics

$$\left\{\begin{aligned}\operatorname{rot}\boldsymbol{E} + \frac{\mu}{\varepsilon}\dot{\boldsymbol{H}} &= 0\\ \operatorname{rot}\boldsymbol{H} - \frac{\varepsilon}{c}\dot{\boldsymbol{E}} &= 0\\ \operatorname{div}\varepsilon\boldsymbol{E} &= 0\\ \operatorname{div}\mu\boldsymbol{H} &= 0\end{aligned}\right\}_{\operatorname{div}\boldsymbol{E} = \operatorname{div}\boldsymbol{H} = 0} \equiv \left\{\begin{aligned}\operatorname{rot}\boldsymbol{E} + \frac{\mu}{c}\dot{\boldsymbol{H}} &= 0\\ \operatorname{rot}\boldsymbol{H} - \frac{\varepsilon}{c}\dot{\boldsymbol{E}} &= 0\end{aligned}\right\}_{\substack{\operatorname{div}\boldsymbol{E} = \operatorname{div}\boldsymbol{H} = 0\\ \boldsymbol{E}\perp\operatorname{grad}\varepsilon\\ \boldsymbol{H}\perp\operatorname{grad}\mu}} \tag{22}$$

has to yield the hydrogen spectrum. Or: the four-component model has to be an approximation to the six-component one.

We will try to represent the electrodynamic model as a solution of source-free electrodynamics (22) by starting from the separation in polar coordinates

$$\Psi = C\,R\,P\,e^{im\varphi}\,e^{i\omega t} = \psi\,e^{i\omega t} \tag{23}$$

or, respectively,

$$\boldsymbol{E}_k = C^{E_k}R^{E_k}P^{m_k^E}\,e^{im_k^E\varphi}\,e^{i\omega t} = E_k\,e^{i\omega t}, \tag{24}$$

$$\boldsymbol{H}_k = C^{H_k}R^{H_k}P^{m_k^H}\,e^{im_k^H\varphi}\,e^{i\omega t} = H_k\,e^{i\omega t}, \tag{25}$$

Here we simply follow the corresponding approaches of the one- and four-component theories. For determining the field components concretely we proceed according to the routines for solving a boundary value problem: we are given covariant source-free electrodynamics (22), that is, eight equations with eight unknowns. We want to find the six field components and the mutual refraction ($\varepsilon_{\text{int}}^{\text{hyd}}$, $\mu_{\text{int}}^{\text{hyd}}$), which shall be called "interfraction" from now on.

The solution must turn out to be integrable, finite, unique and continuous and it must show the hydrogen spectrum. Reducing the problem to its essentials, we face the question:

For what interfraction does electrodynamics yield the hydrogen spectrum?

In order to ensure the autonomy of the solution we are essentially not going to borrow from the preliminary theories. – With one exception: The determination by integration of the hydrogen-interfraction ($\varepsilon_{\text{int}}^{\text{hyd}}$, $\mu_{\text{int}}^{\text{hyd}}$) would be a too much time consuming guessing game, a too lenghtly analytical strain, if we neglected relevant hints from the preliminary theories. Even Schrödinger didn't determine the interfraction by integration, but adopted it from Hamilton's analogy. Dirac got it from Schrödinger, and we borrow it from Dirac by directly reading it off (21).

We still are going to modify the ansatz (23) according to two further aspects. Firstly, we shall consider the well-known form of d'Alembert for the solution,

$$\Psi = f_1(z - v\,t) + f_2(z + v\,t), \tag{26}$$

in order to ensure the greatest generality possible. Secondly, we want to remember that each solution of source-free electrodynamics possesses an axis of rotation, which coincides with the third axis when using the usual spherical polar coordinates [31].

Let us finally have a short look at the basic structure of the two formally identical theories, source-free electrodynamics and relativistic wave mechanics: – How should one comprehend that the dualistic field-and-matter theory, wave mechanics, can be isomorphic to the monistic pure-field theory, source-free electrodynamics? One knows that either monistic, or dualistic, theories can be brought into congruence, if at all, only among each other. For example, it has not been possible to arrange monistic electrodynamics with dualistic general relativity. This has not at all changed after Einstein and Infeld had found cumulative singular solutions of the field equations which should present matter in the form of fields. For it soon became obvious that the solutions were not suited in general for a field description of matter and could not replace the matter parameters of the theory. With this stipulated impossibility of transfering its status from a dualisitc to a monistic one its incompatibility with electrodynamics was established. At a closer look it becomes clear that there does not exist a monism-dualism-gap between electrodynamics and wave mechanics. For wave mechanics – exactly as source-free electrodynamics – is a pure field theory. It gets the point-like reference to particles solely through the Copenhagen interpretation. For all its capacity, which one has to attest to the probabilistic model, doubts have been growing for decades. And unforgotten has been the vehement rejection with which De Broglie, Schrödinger, Lorentz, Einstein, and Von Laue regarded the interpretation.

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